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CERTAIN VARIATIONAL PROBLEMS IN THE GAS DYNAMICS OF AXISYMMETRIC SUPERSONIC FLOW

By U. D. Shmyglevski

ABSTRACT

By using the gas-dynamic functions on the surface bounding the "region of influence," the problem of finding axisymmetric bodies with minimum wave drag has been treated. The method allows the determination of the minimum-drag shape between two prescribed points when the flow properties at the upstream points are known. Application of the method to the determination of minimum-drag body nose and boattail shapes is presented.

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CERTAIN VARIATIONAL PROBLEMS IN THE GAS DYNAMICS OF AXISYMMETRIC SUPERSONIC FLOW*

By U. D. Shmyglevski

An object of many gas dynamics investigations is the finding of bodies that possess the minimum wave resistance. Relatively a long time ago appeared the solution of the linearized equations. For the variational problems of gas dynamics, Nikolski (ref. 1) proposed introducing into consideration the surface bounding the 'region of influence.' Through gas-dynamic functions on such a surface the forces acting on the body can be expressed without integrating the equations of gas dynamics. The first exact solution of the variational gas dynamical problem was obtained by Guderley and Hantsch. In their work (ref. 2), the problem is reduced to the numerical integration of a system of equations.

The problem here considered is a degenerate variational problem. A method of solution of such problem has been worked out by Okhotsimski (ref. 3). The author is deeply grateful to Okhotsimski for his great help in the conduct of this work.

1. The axisymmetric flow of a gas is in cylindrical coordinates determined by the equation of continuity

$$\frac{\partial r\rho w \cos \vartheta}{\partial x} + \frac{\partial r\rho w \sin \vartheta}{\partial r} = 0 \tag{1.1}$$

the equation of motion

$$w \cos \vartheta \frac{\partial w \cos \vartheta}{\partial x} + w \sin \vartheta \frac{\partial w \cos \vartheta}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{1.2}$$

the equation of Bernoulli

$$\frac{w^2}{2} + \frac{\varkappa}{\varkappa - 1}\frac{p}{\rho} = \frac{1}{2}\frac{\varkappa + 1}{\varkappa - 1} \tag{1.3}$$

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and the equation expressing the constancy of the entropy

$$\frac{p}{\rho^{\varkappa}} = \begin{cases} \text{constant in the irrotational case} \\ f(\psi) \text{ in the flow with vorticity} \end{cases} \tag{1.4}$$

where x,r are Cartesian coordinates in the meridional plane of flow, ψ is the stream function, w the velocity referred to the critical flow velocity a_*, ϑ the angle of inclination of the velocity to the flow axis, $\varkappa$ the adiabatic exponent, ρ the density of the gas referred to the density ρ_∞ of the oncoming flow, and p the pressure referred to the product $\rho_\infty a_*^2$.

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Along the characteristics of the system of equations defining the irrotational flow the following relations are satisfied:

First family:

$$dr = \tan\,(\vartheta + \alpha)\,dx, \qquad d\vartheta + \frac{1 + \cos 2\alpha}{\varkappa - \cos 2\alpha}\,d\alpha + \frac{\sin\vartheta \sin\alpha}{r \sin\,(\vartheta + \alpha)}\,dr = 0 \tag{1.5}$$

Second family:

$$dr = \tan\,(\vartheta - \alpha)dx, \qquad d\vartheta - \frac{1 + \cos 2\alpha}{\varkappa - \cos 2\alpha}\,d\alpha - \frac{\sin\vartheta \sin\alpha}{r \sin\,(\vartheta - \alpha)}\,dr = 0 \tag{1.6}$$

where α denotes the Mach angle determined by the relation

$$\sin^2\alpha = \frac{\varkappa p}{\rho w^2}$$

Along the streamlines the relation is satisfied

$$dr = \tan\vartheta\,dx \tag{1.7}$$

Equation (1.1) permits introducing the stream function by the formula

$$d\psi = r\rho w(\cos\vartheta\,ds - \sin\vartheta\,dx) \tag{1.8}$$

2. Let us consider the region of flow of a gas about a body of revolution (fig. 1). Let the gas flow in the direction from the point A to the point B and let the preceding part of the flow be known; that is, the characteristic of the first family AE is known. We must find the generatrix of the body AB that assures the minimum local resistance.

Through the point B is drawn a characteristic of the second family up to the intersection with the characteristic AE at the point C. The region ABC, bounded by the characteristics and the body, will be denoted as the region S.

We transform equation (1.2) with the aid of equation (1.1) to the form

$$\frac{\partial}{\partial x} r(p + \rho w^2 \cos^2 \vartheta) + \frac{\partial}{\partial r} r\rho w^2 \sin \vartheta \cos \vartheta = 0$$

We integrate both sides of this equation over the region S and by Green's formula we pass to the integral over the contour L bounding the region S; we obtain

$$\iint_S \left[\frac{\partial}{\partial x} r(p + \rho w^2 \cos^2 \vartheta) + \frac{\partial}{\partial r} r\rho w^2 \sin \vartheta \cos \vartheta\right] dx\, dr$$

$$= \oint_L - r\rho w^2 \sin \vartheta \cos \vartheta\, dx + r(p + \rho w^2 \cos^2 \vartheta) dr = 0 \tag{2.1}$$

The contour integral (2.1) consists of the integrals over the generatrix AB, the characteristic AC, and the characteristic BC. In each of these integrals we eliminate dx, using respectively equations (1.7), (1.5), and (1.6). The first of the integrals χ is equal to

$$\chi = \int_{r=r_A}^{r_B} pr\, dr \tag{2.2}$$

and differs from the wave resistance of the part of the body considered only by a constant factor.

To determine the shape of the body giving the least wave resistance it would be natural to seek to obtain the minimum of the function (2.2) directly. But for solving such problem it is necessary to know the dependence of p on the form of the generatrix AB, and this can be obtained only by solving the general problem of the flow. Since this solution is unknown, it is necessary to express χ through the integrals over AC and BC with the aid of equation (2.1). This permits solving the problem without recourse to attempts to seek the general integral of the equations of the axially symmetric flow of a gas.

The magnitudes w and ρ are connected with α and the stagnation density ρ_0 by the formulas

$$w^2 = \frac{\varkappa + 1}{\varkappa - \cos 2\alpha}, \qquad \rho = \rho_0\left(\frac{1 - \cos 2\alpha}{\varkappa - \cos 2\alpha}\right)^{\frac{1}{\varkappa-1}}$$

Using these equations in transforming the integral (2.1), we arrive at the desired form for X:

$$X = \rho_0 \sqrt{\frac{\varkappa + 1}{2}} \left\{\int_{r=r_A}^{r_C} \sigma(\alpha)\tau(\alpha) \left[\frac{\sin \alpha}{\varkappa} + \frac{\cos \vartheta}{\sin(\vartheta + \alpha)}\right] r\, dr - \right.$$

$$\left. - \int_{r=r_B}^{r_C} \sigma(\alpha)\tau(\alpha) \left[\frac{\sin \alpha}{\varkappa} - \frac{\cos \theta}{\sin(\vartheta - \alpha)}\right] r\, dr\right\} \tag{2.3}$$

where

$$\sigma(\alpha) = \sqrt{\frac{\varkappa + 1}{\varkappa - \cos 2\alpha}}, \qquad \tau(\alpha) = \left(\frac{1 - \cos 2\alpha}{\varkappa - \cos 2\alpha}\right)^{\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}}$$

We shall express the length $X = x_B - x_A$ and the total discharge of the gas $\Psi = 0$ through the characteristics AC and BC in terms of the integrals over these contours.

From the first relations of (1.5) and (1.6), we readily obtain

$$X = \int_{r=r_A}^{r_C} \cot(\vartheta + \alpha) dr - \int_{r=r_B}^{r_C} \cot(\vartheta - \alpha) dr \tag{2.4}$$

Equation (1.8) permits obtaining

$$\Psi = 0 = \rho_0 \sqrt{\frac{\varkappa + 1}{2}} \left[\int_{r=r_A}^{r_C} \frac{\tau(\alpha) r\, dr}{\sin(\vartheta + \alpha)} + \int_{r=r_B}^{r_C} \frac{\tau(\alpha) r\, dr}{\sin(\vartheta - \alpha)}\right] \tag{2.5}$$

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It is to be noted that the first of the integrals entering (2.3), (2.4), and (2.5) are taken along the known characteristic AC and are therefore functions only of the first limit. The second integrals in the same equations are taken along the unknown characteristic BC. The functions $\alpha(r)$ and $\vartheta(r)$ entering these integrals are to be determined, while the integrals themselves are functionals.

3. In the second relation of (1.6) in the characteristic of the second family, there enters the combination to be integrated, which we denote by

$$d\beta = d\vartheta - \frac{1 + \cos 2\alpha}{\varkappa - \cos 2\alpha}\, d\alpha$$

Integrating this equation we obtain

$$\beta = \vartheta + f(\alpha), \qquad f(\alpha) = \sqrt{\frac{\varkappa + 1}{\varkappa - 1}} \operatorname{arc\,tan} \left(\sqrt{\frac{\varkappa - 1}{\varkappa + 1}} \cot \alpha\right) + \alpha + \text{const} \tag{3.1}$$

For what follows, it is more convenient instead of the unknowns α, ϑ in the characteristic of the second family to introduce the unknowns α, β making use of equation (3.1).

Let us rewrite equations (2.3) to (2.5) and (1.6) in the new notations omitting certain constant factors:

$$\overline{X} = F_1(r_C) - \int_{r=r_B}^{r_C} \Phi_1(r,\alpha,\beta)\,dr \tag{3.2}$$

$$X = F_2(r_C) - \int_{r=r_B}^{r_C} \Phi_2(\alpha,\beta)\,dr \tag{3.3}$$

$$\overline{Y} = 0 = F_3(r_C) + \int_{r=r_B}^{r_C} \Phi_3(r,\alpha,\beta)\,dr \tag{3.4}$$

$$\Phi_4\left(r,\alpha,\beta,\, \frac{d\beta}{dr}\right) = 0 \tag{3.5}$$

where

$$\left.\begin{aligned}
F_1 &= \int_{R=r_A}^{r_C} \sigma(A)\tau(A)\left[\frac{\sin A}{\varkappa} + \frac{\cos\theta}{\sin(\theta + A)}\,R\right]dR \\
F_2 &= \int_{R=r_A}^{r_C} \cot(\theta + A)\,dR \\
F_3 &= \int_{R=r_A}^{r_C} \frac{\tau(A)R\;dR}{\sin(\theta + A)}
\end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned}
\Phi_1 &= \sigma(\alpha)\tau(\alpha)\left\{\frac{\sin\alpha}{\varkappa} - \frac{\cos[\beta - f(\alpha)]}{\sin[\beta - f(\alpha) - \alpha]}\right\}r \\
\Phi_2 &= \cot[\beta - f(\alpha) - \alpha] \\
\Phi_3 &= \frac{\tau(\alpha)r}{\sin[\beta - f(\alpha) - \alpha]} \\
\Phi_4 &= \frac{d\beta}{dr} - \frac{\sin[\beta - f(\alpha)]\sin\alpha}{r\sin[\beta - f(\alpha) - \alpha]}
\end{aligned}\right\} \tag{3.7}$$

where R is the radius, A the Mach angle, and θ the angle of inclination of the velocity to the characteristic AC.

We shall formulate the arising general variational problem of Lagrange with isoperimetric conditions: For given constant r_A, r_B, and X and functions $A(R)$ and $\theta(R)$ entering equations (3.6) to find functions $\alpha(r)$ and $\beta(r)$ that render the difference (3.2) an extremum for the isoperimetric conditions (3.3) and (3.4) and that satisfy the differential condition (3.5).

The class of admissible functions will in what follows be determined in considering the gas-dynamic properties of the problem.

There are considered here flows without shock waves within the region ABC (fig. 1). Hence there arises at once the requirement of continuity of the functions $\alpha(r)$ and $\beta(r)$. Moreover, the following conditions must be satisfied:

$$\alpha(r_C) = A(r_C), \qquad \vartheta(r_C) = \theta(r_C) \tag{3.8}$$

which express the continuity of the Mach angle and the inclination of the velocity at the point C.

Remark. - We note first of all that the problem under consideration is a degenerate one. In fact, the functions Φ_1, Φ_2, Φ_3, and Φ_4 do not contain the derivative $d\alpha/dr$, while they contain the derivative $d\beta/dr$ linearly. This leads generally to the nonsolubility of the variational problem in the classical sense. The solution of the problem completely or partially may not coincide with the integral of the system of Euler equations; and, instead of the classical extremum, for which the first variation of the functional is equal to zero, there may arise a border extremum determined by the physical boundaries of the region of admissible functions.

4. We shall seek the variation $\delta\overline{X}$ in agreement with equation (3.5) and conditions (3.3), (3.4), and (3.8). We construct the sum

$$J = F(r_C) + \int_{r=r_B}^{r_C} \Phi\left(r,\alpha,\beta, \frac{d\beta}{dr},\nu\right) dr \tag{4.1}$$

where

$$F(r_C) = F_1(r_C) + \lambda F_2(r_C) - \mu F_3(r_C)$$

$$\Phi\left(r,\alpha,\beta, \frac{d\beta}{dr},\nu\right) = -\left[\Phi_1(r,\alpha,\beta) + \lambda\Phi_2(\alpha,\beta) + \mu\Phi_3(r,\alpha,\beta) + \nu(r)\,\Phi_4\left(r,\alpha,\beta\ \frac{d\beta}{dr}\right)\right]$$

$\nu(r)$ being variable and λ and μ constant Lagrange multipliers to be determined.

The problem arises of seeking the system of functions $\alpha(r)$, $\beta(r)$ that render the sum (4.1) an unconditional extremum and satisfy the conditions (3.8).

Under the condition $\Phi_4(r,\alpha,\beta,d\beta/dr) = 0$, the sum J differs from the sum $\overline{X}$ by a constant magnitude, since $X = \text{const}$, $\Upsilon = 0$. If these conditions are satisfied on effecting the variation, the variation δJ coincides with the variation $\delta\overline{X}$, since $\delta\Phi_4 = \delta X = \delta\Upsilon = 0$.

In taking the variation of the sum (4.1), it is necessary to take into account that the magnitude r_B is given and therefore $\delta r_B = 0$.

Moreover, the variations δr_C and $\delta\beta_C$ are connected, since the characteristic AC is given. Finally, integrating by parts the expression containing the derivative of the variation $\delta\beta$, we obtain

$$\delta J = \left\{\frac{dF}{dr_C} + \Phi + \nu\left[\left(\frac{d\beta}{dr}\right)_1 - \left(\frac{d\beta}{dr}\right)_2\right]\right\}_{r=r_C} \delta r_{C1} - \nu_B\delta\beta_B + \int_{r=r_B}^{r_C}\left[\Phi_\alpha\delta\alpha + \left(\Phi_\beta - \frac{d}{dr}\Phi_{\beta'}\right)\delta\beta\right]dr \tag{4.2}$$

where the subscripts 1 and 2 denote the derivatives and variations taken respectively along the characteristics of the first and second families.

If conditions (3.3) to (3.5) are identically satisfied, the variations δJ and $\delta\overline{\chi}$ agree for any values of $\nu(r)$. We shall choose this function in such manner that in formula (4.2) the expression with $\delta\beta$ drops out. We set the terms in front of the integral equal to zero; this gives the conditions

$$\left\{\frac{dF}{dr_C} + \Phi + \nu\left[\left(\frac{d\beta}{dr}\right)_1 - \left(\frac{d\beta}{dr}\right)_2\right]\right\}_{r=r_C} = 0, \qquad \nu(r_B) = 0 \tag{4.3}$$

and the equation for the determination of $\nu(r)$,

$$\Phi_\beta - \frac{d}{dr}\Phi_{\beta'} = 0 \tag{4.4}$$

which must be satisfied identically on the entire characteristic BC.

The variation of the sum (4.1) assumes the following form:

$$\delta\overline{\chi} = \int_{r=r_B}^{r_C}\Phi_\alpha\delta\alpha\, dr \tag{4.5}$$

Conditions (4.3) together with (3.8) become the boundary conditions for determining the functions $\alpha(r)$, $\beta(r)$, and $\nu(r)$.

To obtain the classical extremum, the expression before the variation $\delta\alpha$ in formula (4.5) must be equated to zero. Together with (4.4) and (3.5), this will give the system of equations.

5. The system of equations $\Phi_\alpha = 0$, (4.4), and (3.5) written out more fully has the form

$$\sigma(\alpha)\tau(\alpha)\left[\frac{\varkappa + 1}{2}\,\frac{\sin 2(\beta - f)}{\sin \alpha} - 2(1 + \cos 2\alpha)\cos \alpha\right] + \lambda \frac{\omega(\alpha)}{r}$$
$$+ \mu\tau(\alpha)[\omega(\alpha)\cos(\beta - f - \alpha) - (\varkappa + 1)\cot \alpha \sin(\beta - f - \alpha)]$$
$$- \frac{\nu}{2r^2}\,[\sin^2 2\alpha - 2(\varkappa - \cos 2\alpha)\sin^2(\beta - f)] = 0 \tag{5.1}$$

$$\frac{d\nu}{dr} = \frac{1}{\sin^2(\beta - f - \alpha)}\left\{r\tau(\alpha)[\sigma(\alpha)\cos \alpha - \mu \cos(\beta - f - \alpha)] - \lambda + \frac{\nu}{r}\sin^2 \alpha\right\} \tag{5.2}$$

$$\frac{d\beta}{dr} = \frac{\sin(\beta - f)\sin \alpha}{r \sin(\beta - f - \alpha)} \tag{5.3}$$

where

$$\omega(\alpha) = 1 - \varkappa + 2 \cos 2\alpha$$

The fact should be observed that equation (5.1) is not a differential equation.

Remark. - The equations of the first order (5.2) and (5.3) give two arbitrary factors in determining the functions. Moreover, r_C and the values of the constants λ and μ are still not determined. In all, there are five arbitrary factors in determining the functions.

To obtain the extremum in the classical sense, the required functions must be subjected to the boundary conditions (3.8) and (4.3) and the isoperimetric conditions (3.3) and (3.4). Altogether this gives six conditions.

From this it becomes evident that the problem is insoluble if the condition is imposed that the required functions must satisfy equations (5.1) to (5.3) over the entire characteristic BC.

6. Equation (5.3) connects the functions $\alpha(r)$ and $\beta(r)$. If the system of boundary conditions is complete, one of the functions completely determines the other. Let us consider the function $\alpha(r)$ on the characteristic BC and for this let us turn to figure 2. The shape of the curve $\alpha(r)$ plays no part in the further considerations. For simplicity we show in figure 2 one of the forms of behavior of $\alpha(r)$. The function $\alpha(r)$ on the characteristic AE is represented by the curve ae.

The theory considered is suitable only for supersonic flows. Hence, the upper natural boundary of the region of variation of α is the line $\alpha = \pi/2$.

The lower boundary of the region can be determined by considering the gas-dynamic properties of the problem. All possible functions $\alpha(r)$ on the characteristic BC can be obtained only by a change of the flow boundary, the generatrix of the body AB (fig. 1). We must emphasize that gas flows are considered that have only one boundary AB and that do not contain shock waves in the triangle ABC.

To each value $r = r_C$ on the given characteristic AE corresponds a completely determined value $\alpha = \alpha_C$. That flow of the gas must be found for which the curve $\alpha(r)$ on the characteristic DC will be situated below all other possible curves. Physically, it is evident that the greatest expansion of the flow is assured by the presence of a break of the generatrix AB at the point A. In this case (refs. 4 and 5), the characteristics of the first family diverge as a bundle of lines from the point A (fig. 1) up to a certain characteristic AD, on which, at the point A, the angle of inclination of the velocity coincides with the angle of the tangent to the contour AB at the point A. The characteristic AD is a line of weak discontinuity in the region of flow. In figure 2 let the relation $\alpha = \alpha_*(r)$ on the characteristic of the second family CD of the flow be represented by the curve cd. The characteristic CB then consists of the segment CD not coinciding with the extremal and the extremal segment DB.

We shall call a function $\alpha(r)$ admissible if it is a continuous function satisfying the inequality $\alpha_*(r) \leq \alpha(r) \leq \pi/2$. The admissible functions $\beta(r)$ are then also completely determined.

7. Taking the previously explained properties of the characteristic BC into account, we shall derive the boundary conditions for the segment BD of the characteristic (fig. 1).

We write the sum (4.1) in the form

$$J = F(r_C) + \int_{r=r_B}^{r_D} \Phi \, dr + \int_{r=r_D}^{r_C} \Phi \, dr \tag{7.1}$$

Let us consider the second integral of (7.1). In varying the upper limit, only variations along the given characteristic AE are permitted. The variation of the lower limit is broken up into two. It is admissible in the first place to vary r_D along the characteristic of the second family; that is, to vary the region occupied by the bundle of characteristics. In the second place, it is permissible to vary r_D along the characteristic of the second family, but this variation must not now be carried out independently of the variation of r_C, because with r_C given the entire characteristic DC is determined. In connection with this it is more convenient to vary the second integral of (7.1) as a whole, as a function of r_C, not forgetting of course also the variation along the characteristic of the second family. The variation of this integral is easily obtained by considering an integral of the type (7.1) over the contour ADC for fixed characteristics AC and AD. The integral over DC is then expressed through the integrals over AC and AD.

It is necessary also to bear in mind that on BC all functions are continuous. From (3.5) the continuity of the derivative $d\beta/dr$ then also follows.

Carrying out all the required calculations and taking into account the continuity of α, β, and $d\beta/dr$, we arrive at an expression for δJ agreeing with (4.2), if in the latter we replace the point C by the point D. The arbitrariness in the choice of the characteristic AD renders the problem soluble.

The system of equations for constructing the characteristic BD remains unchanged and agrees with (5.1) to (5.3).

Equating the expressions before the integral in the expression for δJ to zero we obtain

$$\left\{\frac{dF(r_D)}{dr_D} + \Phi + \nu\left[\left(\frac{d\beta}{dr}\right)_1 - \left(\frac{d\beta}{dr}\right)_2\right]\right\}_{r=r_D} = 0 \qquad (7.2)$$

$$\nu(r_B) = 0 \qquad (7.3)$$

Equations (3.3), (3.4), (3.8), (7.2), and (7.3) form a complete system of boundary conditions for the solution of the problem.

8. On the characteristic BD, assuring a minimum of the resistance (fig. 1), equations (5.1) to (5.3) must be satisfied for the boundary conditions (3.8) and (7.2) at the point D and the condition (7.3) at the point B. We introduce an arbitrary characteristic of the first family GF. The segment of the generatrix GB has no effect on the flow to the left of GF. Therefore the part of the generatrix GB should possess minimum resistance for the given characteristic GF and the points G and B (otherwise, a decrease of the resistance of the segment GB decreases the resistance of the entire generatrix AB). On the segment of the characteristic FB, equations (5.1) to (5.3) are satisfied, and at point B the condition (7.3). Hence, at the point F the transversality condition (7.2) written for $r = r_F$ must be satisfied. This condition, by

virtue of the arbitrariness of choice of the characteristic GF, is satisfied also on the entire characteristic BD. Hence, it must be an integral of the system (5.1) to (5.3). In expanded form, (7.2) is given by

$$\tau(\alpha)r\left[\frac{1}{\sin(\vartheta+\alpha)}+\frac{1}{\sin(\vartheta-\alpha)}\right][\mu-\sigma(\alpha)\cos\vartheta]$$

$$-\lambda[\cot(\vartheta+\alpha)-\cot(\vartheta-\alpha)]+\nu\left[\left(\frac{d\beta}{dr}\right)_1-\frac{\sin\vartheta\sin\alpha}{r\sin(\vartheta-\alpha)}\right]=0 \tag{8.1}$$

From equations (5.1), (5.2), and (8.1), we eliminate λ and μ. As a result we obtain a linear homogeneous differential equation of the first order for determining $\nu(r)$. Recalling condition (7.3), we at once conclude that

$$\nu(r)\equiv 0 \tag{8.2}$$

The two obtained integrals of the system of equations satisfy the boundary conditions and together with (5.1) permit finding the expressions for $\alpha(r)$ and $\vartheta(r)$. In place of equation (5.1) it is simpler to make use of equation (5.2), substituting in it $\nu = 0$:

$$r\tau(\alpha)[\sigma(\alpha)\cos\alpha-\mu\cos(\vartheta-\alpha)]=\lambda \tag{8.3}$$

Eliminating λ from (8.1) and (8.3) gives

$$\sqrt{\frac{\varkappa+1}{\varkappa-\cos 2\alpha}}\,\frac{\cos(\vartheta+\alpha)}{\cos\alpha}=\mu \tag{8.4}$$

Substitution of ϑ from (8.4) in (8.3) gives the equation connecting α and r:

$$r\left(\frac{1-\cos 2\alpha}{\varkappa-\cos 2\alpha}\right)^{\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}}\sqrt{\frac{\varkappa-\cos 2\alpha}{\varkappa+1}}\left[\frac{(\varkappa+1)\cos\alpha}{\varkappa-\cos 2\alpha}\right.$$

$$\left.-\mu^2\cos\alpha\cos 2\alpha-\mu\sin 2\alpha\sqrt{\frac{\varkappa+1}{\varkappa-\cos 2\alpha}-\mu^2\cos^2\alpha}\right]=|\lambda| \tag{8.5}$$

Finally, eliminating μ from (8.1) and (8.3), we obtain the equation

$$r\left(\frac{1 - \cos 2\alpha}{\varkappa - \cos 2\alpha}\right)^{\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}} \sqrt{\frac{\varkappa + 1}{\varkappa - \cos 2\alpha}} \frac{\sin^2 \vartheta}{\cos \alpha} = \lambda \tag{8.6}$$

which permits writing down the expression for $\vartheta(\alpha)$:

$$\vartheta = \operatorname{sign} \vartheta_D \left| \arcsin \left[\sqrt{\frac{|\lambda| \cos \alpha}{r}} \left(\frac{1 - \cos 2\alpha}{\varkappa - \cos 2\alpha}\right)^{-\frac{1}{4}\frac{\varkappa+1}{\varkappa-1}} \left(\frac{\varkappa - \cos 2\alpha}{\varkappa + 1}\right)^{\frac{1}{4}} \right] \right| \tag{8.7}$$

In this formula there is attached to the magnitude ϑ the symbol ϑ_D in accordance with the following considerations. The magnitude λ is constant; if the magnitudes $\alpha \neq \pi/2$ and $\vartheta_D \neq 0$, the magnitude ϑ is not equal to zero anywhere on the characteristic BD, as follows from (8.6) and, being a continuous function, does not change sign. If $\vartheta = 0$ at one point, then $\vartheta \equiv 0$ on the entire characteristic BD.

The magnitudes x and ψ on the characteristic BD are readily expressed in quadratures with the aid of equations (1.6) and (1.8).

The obtained formulas actually give the solution of the problem. This may be confirmed directly by a check.

The magnitudes λ and μ are obtained from the known r, α, and ϑ at any point from formulas (8.6) and (8.4).

It is of interest to observe that any streamline HF of the obtained field of motion of the gas (fig. 1) is an extremal for a given characteristic AD and the points H and F, since on the characteristic DF equations (5.1) to (5.3) and all boundary conditions are satisfied.

The generatrix AB is obtained as solution of the problem of Goursat between the characteristic AD and the now known characteristic BD.

9. Up to now we have considered the required conditions for an extremum. In the problem under consideration the minimum of the sum $\overline{X}$ must be obtained.

Again we turn to figure 2 and recall expression (4.5) for $\delta\overline{X}$.

Let there be found a solution satisfying equations (5.1) to (5.3) and all boundary conditions. By virtue of equation (5.1), the equation $\Phi_\alpha = 0$ is satisfied on the entire curve bd; on the curve cd, generally speaking, $\Phi_\alpha \neq 0$. The minimum of the sum $\overline{X}$ will occur in the case where $\Phi_\alpha > 0$ for $\delta\alpha > 0$ and $\Phi_\alpha < 0$ for $\delta\alpha < 0$. In fact, for $\delta\alpha > 0$ on the curve bd, we then have $\delta\overline{X} > 0$, for $\delta\alpha < 0$ we have $\delta\overline{X} > 0$, as follows from equation (4.5). Moreover, on the curve cd in this case $\Phi_\alpha \geq 0$. The contrary case is impossible, since on the curve cd the least of all possible values of $\alpha(r)$ occurs. By virtue of this property, the curve cd is admissible to the variation on it of $\delta\alpha > 0$. From $\Phi_\alpha \geq 0$ and (4.5), it follows that these permissible variations lead to $\delta\overline{X} > 0$. There occurs here a border extremum - that is, existing only because of the positiveness of the admissible variations $\delta\alpha$.

Let us check the possibility of such variation of the obtained curve cdb for which the sum X decreases. On the segment cd with the exception of the point d, the admissible variations $\delta\alpha > 0$ lead, as has been explained, to $\delta\overline{X} > 0$ by virtue of $\Phi_\alpha > 0$. The sign of Φ_α can be checked for example by the direct computation of Φ_α on cd. At the point d every infinitely small element of the curve $\alpha(r)$ not coinciding in direction with db leads to $\delta\overline{X} > 0$, if $\Phi_{\alpha\alpha} > 0$. In fact, by virtue of equation (3.5), the variation $\delta\beta$ has a higher order of smallness than the variation $\delta\alpha$; hence, the sign of the magnitude Φ_α on the element of the curve considered is conditioned by the sign of $\Phi_{\alpha\alpha}$. Thus, the minimum $\overline{X}$ is assured by the coincidence of this element with an element of the curve db. For the following point the same considerations hold true. By passing from one point to the next we can see that the entire curve cdb assures a minimum $\overline{X}$.

It is necessary also to check the sense of the extremum from the boundary conditions.

If the boundary conditions (7.2) and (7.3), on satisfying conditions (3.3), (3.4), and (3.8) and the equations of the problem, give the minimal value $\overline{X}_{min}$, then we must have

$$\overline{X} > \overline{X}_{min} \qquad \text{for} \qquad \nu_B \neq 0 \tag{9.1}$$

with observance of conditions (7.2), (3.3), (3.4), (3.8), and equations (5.1) to (5.3), and

$$\overline{X} > \overline{X}_{min} \tag{9.2}$$

for

$$\left\{\frac{dF(r_D)}{dr_D} + \Phi + \nu\left[\left(\frac{d\beta}{dr}\right)_1 - \left(\frac{d\beta}{dr}\right)_2\right]\right\}_{r=r_D} \neq 0$$

with observance of conditions (7.3), (3.3), (3.4), (3.8), and equations (5.1) to (5.3).

The inequalities (9.1) and (9.2) can likewise be checked on carrying out the computations. In both cases of course all the conditions must be satisfied only in choosing some new point D in the family of characteristics.

10. Let us consider the problem of the nozzle. This problem is solved in an entirely similar manner. It has been considered in detail and reduced to ordinary differential equations in the work of reference 2. In this case, too, we shall indicate the solution in finite form. The functions $\alpha(r)$ and $\vartheta(r)$ on the characteristic BD (fig. 3) are from the same equations (8.5) and (8.7). The tables of gas-dynamic functions in the region AOCD for the case of a two-dimensional transition surface are given in reference 6.

11. In the case of vortical flow, it is also possible to obtain simple relations on the required characteristic. As independent variable in this case the stream function ψ must be chosen.

On the required characteristic AC there must be given the entropy function

$$\varphi(\psi) = p^{\frac{1}{\varkappa-1}} \rho^{-\frac{\varkappa}{\varkappa-1}}$$

The problem of determining a body having minimum resistance reduces for given functions $A(\psi)$, $\theta(\psi)$, $\Phi(\psi)$, $R(\psi)$, and magnitudes X, r_A, r_B to determining the functions $\alpha(\psi)$, $\vartheta(\psi)$, $r(\psi)$, rendering the functional

$$\bar{X} = \int_{\psi=0}^{\psi_C} \left\{ \sqrt{\frac{\varkappa + 1}{\varkappa - \cos 2A}} \left[\cos \theta + \frac{1}{\varkappa} \sin A \sin(\theta + A) \right] - \sqrt{\frac{\varkappa + 1}{\varkappa - \cos 2\alpha}} \left[\cos \vartheta - \frac{1}{\varkappa} \sin \alpha \sin(\vartheta - \alpha) \right] \right\} d\psi$$

minimum for the isometric condition

$$X = \int_{\psi=0}^{\psi_c} \frac{\varphi(\psi)}{\sqrt{\varkappa}} \left[\frac{\cos(\theta + A)}{R} \left(\frac{\varkappa + 1}{2\varkappa} \frac{1 - \cos 2A}{\varkappa - \cos 2A} \right)^{-\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}} + \frac{\cos(\vartheta - \alpha)}{r} \left(\frac{\varkappa + 1}{2\varkappa} \frac{1 - \cos 2\alpha}{\varkappa - \cos 2\alpha} \right)^{-\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}} \right] d\psi$$

and the two differential conditions

$$\left.\begin{aligned} &\frac{dr}{d\psi} + \frac{\varphi(\psi)}{\sqrt{\varkappa} r} \left(\frac{\varkappa + 1}{2\varkappa} \frac{1 - \cos 2\alpha}{\varkappa - \cos 2\alpha} \right)^{-\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}} \sin(\vartheta - \alpha) = 0 \\ &\frac{d\vartheta}{d\psi} - \frac{1 + \cos 2\alpha}{\varkappa - \cos 2\alpha} \frac{d\alpha}{d\psi} - \frac{\sin \vartheta \sin \alpha}{r \sin(\vartheta - \alpha)} \frac{dr}{d\psi} + \frac{\sin 2\alpha}{2\varkappa} \frac{d \ln \varphi}{d\psi} = 0 \end{aligned}\right\} \quad (11.1)$$

Without repeating the considerations, analogous to the preceding, we present the final results.

The required characteristic BC consists of the nonextremal segment CD, which is a characteristic of the limiting rarefaction flow, and the extremal segment BD. On this segment the required functions are connected by the following relations:

$$|\lambda| \left(\sin 2\vartheta + \frac{1}{\varkappa} \sin 2\alpha \right) + \mu(1 - \cos 2\vartheta) = 0$$

$$\frac{|\lambda| \varphi(\psi) \cos \alpha}{\sqrt{\varkappa} r} \left(\frac{\varkappa + 1}{2\varkappa} \frac{1 - \cos 2\alpha}{\varkappa - \cos 2\alpha} \right)^{-\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}} - \sqrt{\frac{\varkappa + 1}{\varkappa - \cos 2\alpha}} \sin^2 \vartheta = 0$$

where λ is a constant and μ a variable Lagrange multiplier. The magnitudes λ and $\mu(\psi_D)$ are determined from the last relations in terms of the known magnitudes α, ϑ, r, and ψ at the point D. The function $r(\psi)$ is determined by the differential equation (11.1), and $\mu(\psi)$ by the equation

$$\frac{d\mu}{d\psi} = - \frac{\varphi(\psi)}{\sqrt{\varkappa} r^2} \left(\frac{\varkappa + 1}{2\varkappa} \frac{1 - \cos 2\alpha}{\varkappa - \cos 2\alpha} \right)^{-\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}} [|\lambda| \cos(\vartheta - \alpha) + \mu \sin(\vartheta - \alpha)]$$

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12. As an illustration of the method, the computed generatrices of bodies of revolution are represented in the figures. Bodies having a given head cone are represented in figures 4 and 5; the half-angle of the cone is equal to 35°. The rear parts of the semi-infinite cylinder are represented in figures 6 and 7. The Mach numbers of the oncoming flow M_∞ and the coefficients of the wave resistance of the bodies c_x are given on the figures.

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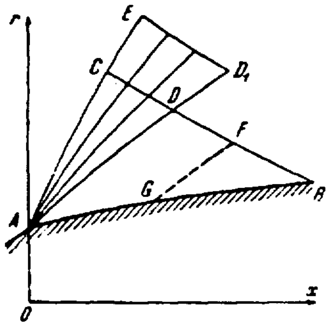

Fig. 1

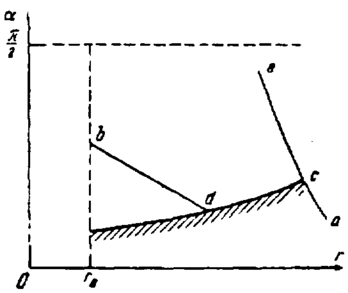

Fig. 2

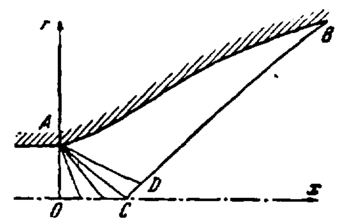

Fig. 3

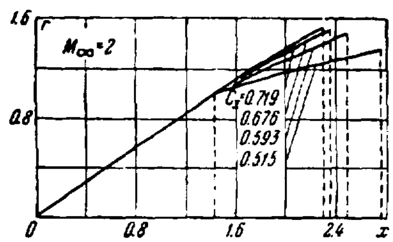

Fig. 4

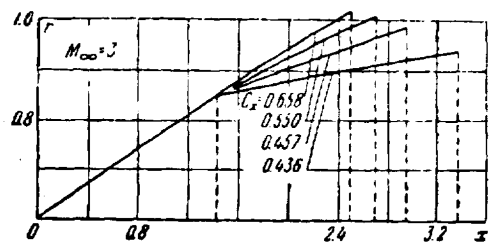

Fig. 5

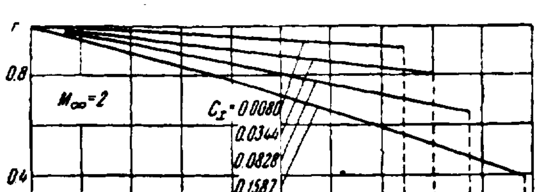

Fig. 6

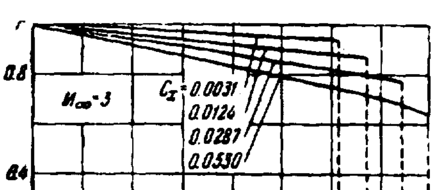

Fig. 7